FINAL EXAM (MATH7431P - USTC, FALL 2025)

Instruction: There are three categories of problems, 100 points in total. The exam time is from **2:00 pm to 5:00 pm, January 19, 2026**. Please follow the instruction of each problem. There is one bonus problem, 10 points; points earned from this bonus problem will be added to your total score. This exam is required to be completed WITHOUT any references.

A. PROBLEMS on DEFINITIONS and STATEMENTS (50 points in total, 5 points for each)

Problem A.1. Write down the definition of a stably framed Hamiltonian structure (ω, λ) on an odd-dimensional manifold M **[3 points]** and its symplectization (in particular, its symplectic structure) **[2 points]**.

Problem A.2. Let (M, J) be an even-dimensional manifold equipped with an almost complex structure and $E \to M$ be a complex vector bundle. Write down the definition of the notation $\Omega^{1,1}(M; E)$.

Problem A.3. Write down the statement of the Sobolev embedding theorem for compact domain $\Omega \subset \mathbb{R}^n$ when $p > n$ **[2 points]**. Please explain (in terms of their definitions) all the notations involved in this statement **[3 points]**.

Problem A.4. Let $\mathbb{D}^2$ be a closed 2-disk in $\mathbb{C}$ and j be a complex structure on $\mathbb{D}^2$. For a differentiable map $u : (\mathbb{D}^2, j) \to (\mathbb{C}^n, J)$, write down the definition of u being J-holomorphic **[2 points]**. Moreover, write down the regularity theorem which implies that when J is smooth, then u is in fact a smooth map (when restricted to a smaller domain inside $\mathbb{D}^2$ if necessary) **[3 points]**.

Problem A.5. Let $D : X \to Y$ be a bounded linear operator between two Banach spaces X, Y. Write down the definition of D being a Fredholm operator and its index $\mathrm{ind}(D)$ **[3 points]**. Explain why if two Fredholm operators D_1, D_2, connected by a continuous path of Fredholm operators, then they have the same indices **[2 points]**.

Problem A.6. State the Carleman Similarity Principle.

Problem A.7. State the Gromov compactness theorem (for the case when the target manifold (M, ω) is closed and the domain Riemann surfaces are those without marked points).

Problem A.8. State the convergence phenomenon in a rigorous way when the limit is a "broken flowline" in the Morse setting, starting from a sequence of gradient flowlines $u_n : \mathbb{R} \to (M, g, F)$, where F is a Morse function and g is a metric on M, with fixed asymptotic ends $\lim_{s\to\pm\infty} u_n(s) = x_\pm \in \mathrm{Crit}(F)$.

Problem A.9. For pointed Riemann surfaces (Σ_g, j, Θ), list all the cases where they are *not* stable.

Problem A.10. Let $u : (\Sigma, j) \to (M, J)$ be a J-holomorphic curve. Write down the definition of u being simple. (Note that do NOT use "somewhere injective" to define simple, even though they are equivalent.)

B. PROBLEMS on COMPUTATIONS (20 points in total, 10 points for each)

Problem B.1. Given a closed symplectic manifold (X, Ω) and a 1-periodic Hamiltonian function $H : S^1 \times X \to \mathbb{R}$. Consider odd-dimensional manifold $M := S^1(t) \times X$. Complete the following problems.

(i) **[4 points]** Prove that

$$(\omega, \lambda) := (\Omega + dt \wedge dH, dt)$$

is a stably framed Hamiltonian structure on M and calculate its Reeb vector field.

(ii) **[6 points]** For any compatible almost complex structure $J \in \mathcal{J}((\omega, \lambda))$ on the symplectization $\mathbb{R}\times M$ (which can be identified with a smoothly S^1-parametrized family of compatible almost complex structures $\{J_t\}_{t\in S^1}$ on (X, ω), due to the translation invariant property), consider a J-holomorphic cylinder

$$u : (\mathbb{R} \times S^1, j) \to (\mathbb{R} \times M, J)(= ((\mathbb{R} \times S^1) \times X, J)).$$

Write $u = (\varphi, \tilde{v})$ where $\varphi : \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ and $\tilde{v} : \mathbb{R} \times S^1 \to X$. Assume that φ is injective, so after a reparametrization one can write $u = (\mathbb{1}, v)$. Write out the express of the partial differential equation that v should satisfy and justify your answer with necessary details.

Problem B.2. Recall that in Hamiltonian Floer homology theory associated to the Hamiltonian system $(M, \omega, H : [0, 1] \times M \to \mathbb{R}, J = \{J_t\}_{t\in\mathbb{R}/\mathbb{Z}})$ where (M, ω) is closed symplectic manifold, a Floer cylinder $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to M$ satisfies

$$\frac{\partial u}{\partial s} + J_t(u)\frac{\partial u}{\partial t} - \nabla H_t(u) = 0$$

where the gradient ∇ (depending on $t \in \mathbb{R}/\mathbb{Z}$) is taken with respect to the metric $\langle \cdot, \cdot \rangle_t := \omega(\cdot, J_t \cdot)$ (equivalently, $\nabla H_t = J_t X_{H_t}$ where X_{H_t} is the Hamiltonian vector field of H on (M, ω)). Prove that if the energy $E(u) < \infty$, then there exist a sequence of real numbers $\{s_n^+\}_{n\in\mathbb{N}}$ diverging to ∞ and a sequence of real numbers $\{s_n^-\}_{n\in\mathbb{N}}$ diverging to $-\infty$, such that the loops $x_n^{\pm} := u(s_n^{\pm}, \cdot)$ converge in C^∞-sense to closed Hamiltonian orbits $x_\pm$ of (M, ω, H, J), respectively.

For Problem B.2, without justifying the C^∞-convergence will **lose 4 points!

C. PROBLEMS on PROOFS (30 points in total, 15 points for each)

Problem C.1. Fix a closed symplectic manifold (M, ω) and an ω-compatible almost complex structure J. Prove that there exist a constant $\hbar > 0$ and $C > 0$ such that for any J-holomorphic curve $u : (\Sigma, j) \to (M, \omega, J)$, where (Σ, j) is a Riemann surface with boundary, if $E(u) < \hbar$, then

$$E(u) \leq C \cdot \mathrm{length}^2_{g_J}(u(\partial\Sigma)).$$

Here, g_J is the induced metric $\omega(\cdot, J\cdot)$ from ω and J.

**For Problem C.1, one needs to state and use the monotonicity lemma directly (no need to prove this lemma). Any other intermediate results/claims need justifications.

Problem C.2. Fix a closed symplectic manifold (M, ω) and an ω-compatible almost complex structure J. Let $u_n : (\Sigma, j) \to (M, \omega, J)$ be a sequence of J-holomorphic curve such that there exists a uniform (independent of n) constant $C > 0$ with $E(u_n) < C$. Prove that if there exists a sequence of points $\{z_n\}_{n\in\mathbb{N}}$ such that z_n converge to z and $|du_n(z_n)| \to +\infty$, then z is a bubble point of the sequence $\{u_n\}_{n\in\mathbb{N}}$.

**For Problem C.2, feel free to use the following two results.

Lemma 0.1. *Let (X, d) be a complete metric space, $\delta > 0$, $x \in X$, and $f : X \to [0, \infty)$ be a continuous function. Then there exists some $\xi \in X$ and $\epsilon > 0$ with the following properties.*

(i) $\epsilon \leq \delta$;
(ii) $d(x, \xi) < 2\delta$;
(iii) $\epsilon f(\xi) \geq \delta f(x)$;

(iv) $2f(\xi) \geq \sup_{B_\epsilon(\xi)} f$;

where $B_\epsilon(\xi)$ is the closed ball centered at ξ with radius ϵ.

Lemma 0.2. *Let $v_n : (B_{R_n}(0), j_{\text{std}}) \to (M, \omega, J)$ be a sequence of J-holomorphic maps (into a closed symplectic manifold (M, ω)), where $R_n \to \infty$. If $E(v_n) < C$ for a uniform upper bound $C > 0$ (independent of n), then there exists a subsequence of $\{v_n\}_{n \in \mathbb{N}}$ converging to a J-holomorphic map $v_\infty : (\mathbb{C}, j_{\text{std}}) \to (M, J)$ in C^∞_{loc}-sense (i.e, smoothly over any compact subset of $\mathbb{C}$).*

BONUS problem (10 points). Denote by $\mathcal{M}_{g,\ell}$ the moduli space of (equivalence classes of) pointed Riemann surfaces with genus g and ℓ marked points. Describe $\overline{\mathcal{M}}_{0,4}$ with enough details (in particular, establish the "boundary" elements in $\overline{\mathcal{M}}_{0,4} \backslash \mathcal{M}_{0,4}$).